

MINIMAL TORI WITH LOW NULLITY

DAVID L. JOHNSON, OSCAR PERDOMO

ABSTRACT. The *nullity* of a minimal submanifold $M \subset S^n$ is the dimension of the nullspace of the second variation of the area functional. That space contains as a subspace the effect of the group of rigid motions $SO(n+1)$ of the ambient space, modulo those motions which preserve M , whose dimension is the *Killing nullity* $kn(M)$ of M . In the case of 2-dimensional tori M in S^3 , there is an additional naturally-defined 2-dimensional subspace; the dimension of the sum of the action of the rigid motions and this space is the *natural nullity* $nnt(M)$. In this paper we will study minimal tori in S^3 with natural nullity less than 8. We construct minimal immersions of the plane \mathbb{R}^2 in S^3 that contain all possible examples of tori with $nnt(M) < 8$. We prove that the examples of Lawson and Hsiang with $kn(M) = 5$ also have $nnt(M) = 5$, and we prove that if the $nnt(M) \leq 6$ then the group of isometries of M is not trivial.

1. INTRODUCTION

Let $\tilde{\rho} : M \rightarrow S^3$ be a minimal immersion of an oriented compact surface without boundary M in the unit three dimensional sphere $S^3 \subset \mathbb{R}^4$. When there is no confusion, we will identify $m \in M$ with $\tilde{\rho}(m)$ and the vectors in $T_m M$ with those in $d\tilde{\rho}_m(T_m M) \subset \mathbb{R}^4$. Let $N : M \rightarrow S^3$ be a Gauss map, i.e. $N(m) \perp T_m M$ and $\langle N(m), m \rangle = 0$. For any $m \in M$, $a(m)$ will denote the nonnegative principal curvature of M at m and $W_1(m)$ and $W_2(m)$ will denote two unit tangent vectors such that $dN_m(W_1(m)) = -a(m)W_1(m)$ and $dN_m(W_2(m)) = a(m)W_2(m)$. Notice that $a(m)$ is uniquely determined but the vectors $W_1(m)$ and $W_2(m)$ are not. When M is a torus, it is known that for every m , $a(m)$ is positive [2], therefore in this case we can choose $W_1(m)$ and $W_2(m)$ so that they define smooth vector field in M . In the following, if M is a torus, W_1 and W_2 will denote such unit tangent vector fields and $a : M \rightarrow \mathbb{R}$ will be the smooth function given by the positive principal curvature. Since we are identifying vectors in $T_m M$ with those in $d\tilde{\rho}_m(T_m M)$ using the homomorphisms $d\tilde{\rho}_m$, the tangent vector fields on M are given by functions $X : M \rightarrow \mathbb{R}^4$ such that $\langle X(m), \tilde{\rho}(m) \rangle = 0$ and $\langle X(m), N(m) \rangle = 0$ for all $m \in M$. Given a fixed, skew-symmetric 4×4 matrix B , define $f_B : M \rightarrow \mathbb{R}$ by $f_B = \langle B\tilde{\rho}(m), N(m) \rangle$. Since M is minimal, M is a critical point of the area functional. The second variation of the area function at this critical point is given by the stability operator

$$J : C^\infty(M) \rightarrow C^\infty(M) \quad \text{given by} \quad J(f) := -\Delta f - 2a^2 f - 2f.$$

It is clear that f_B satisfies the elliptic equation $J(f_B) = 0$ because, when we move the immersion M by the group of isometries $e^{Bt} : S^3 \rightarrow S^3$ we induce a family that leaves the area and second fundamental form constant; f_B is the function associated with this family. The *nullity* of a minimal surface is defined as the dimension of the kernel of the operator J and will be denoted by $n(M)$. In [3], Lawson and Hsiang classify all the minimal surfaces that are invariant under a 1-parametric group of isometries in S^3 . As they point out at the end of their paper, one way to see this classification is the following: Define

$$KS = \{f_B : B \in so(4)\}; \quad \text{Killing nullity} = \dim(KS) = kn(M).$$

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We have that $kn(M) \leq n(M)$ and in general the Killing nullity is expected to be 6 since the dimension of $so(4)$ is 6. Then, they classify all the examples of surfaces with $kn(M) < 6$. These examples turn out to be the totally geodesic spheres with Killing nullity 3, the Clifford tori with Killing nullity 4 and a collection of tori with Killing nullity 5. It is known that when M is a torus, for any angle θ , the function $h_\theta : M \rightarrow \mathbb{R}$ given by

$$h_\theta = \cos(\theta)a^{-\frac{3}{2}}W_1(a) + \sin(\theta)a^{-\frac{3}{2}}W_2(a)$$

satisfies that $J(h_\theta) = 0$. Actually, this equation was the starting point in the classification of all constant mean curvature surface in \mathbb{R}^3 and all minimal tori in S^3 given by Pinkall and Sterling in [5]. The study of the nullity of minimal tori will be helpful in understanding their examples. When M is a torus, we can define the following space and the following integer

$$NS = \{f_B + \lambda h_\theta : B \in so(4), \lambda, \theta \in \mathbb{R}, \}, \quad \text{Natural nullity for tori} := \dim(NS) := nnt(M).$$

Clearly NS is a subset of the kernel of J and therefore $nnt(M) \leq n(M)$. The natural nullity is expected to be 8 because of the 6 parameters of $so(4)$ and the two parameters λ and θ in the definition of the space NS .

In this paper we study minimal tori with natural nullity for tori less than 8. We construct, in Theorem (3.11), minimal immersions of the plane \mathbb{R}^2 in S^3 that contain *all* possible examples of tori with $nnt(M) < 8$, which is shown in Theorem (3.12). We prove (Corollary (5.8)) that the examples of Lawson and Hsiang with $kn(M) = 5$ are the only immersed tori satisfying $nnt(M) = 5$, although the question of the (total) nullity of these examples is not resolved. Finally, we show in Theorem (5.11) that if $nnt(M) \leq 6$ then the group of isometries is not trivial.

2. PRELIMINARIES

In this section we will review some known results that will be used later on. The first result has already been used in the introduction in order to define the unit tangent smooth vector fields W_1 and W_2 in an immersed minimal torus of S^3 .

Theorem 2.1. [Lawson, [2]] *If $M \subset S^3$ is a closed minimal surface and $a : M \rightarrow \mathbb{R}$ denotes the nonnegative principal curvature function, then a is positive everywhere if and only if $\chi(M) = 0$.*

The next theorem also was used in the introduction in order to defined define the natural nullity for tori. Even though it is a known result, for completeness sake we will provide a proof at the end of this section.

Theorem 2.2. *If $M \subset S^3$ is a minimal immersed torus, and $W_1 : M \rightarrow S^3$ and $W_2 : M \rightarrow S^3$ are unit vector field that define the principal directions, then the functions*

$$h_0, h_{\frac{\pi}{2}} : M \rightarrow \mathbb{R} \text{ given by } h_0(m) = a^{-\frac{3}{2}}W_1(a) \text{ and } h_{\frac{\pi}{2}} = a^{-\frac{3}{2}}W_2(a)$$

satisfy

$$J(h_0) = -\Delta h_0 - 2h_0 - 2a^2 h_0 = 0 = J(h_{\frac{\pi}{2}}).$$

The following theorem will be used in section 4 to prove that one subfamily of the family of examples of minimal immersion of the plane in S^3 we have constructed corresponds to the Lawson-Hsiang examples.

Theorem 2.3. [Ramanaham [6]] *Let $\tilde{\rho} : M \rightarrow S^3$ be a minimal immersion from an oriented compact surface. Suppose that M admits a one parameter group of isometries $\phi_t : M \rightarrow M$ with respect to the induced metric. Then, there exists a one-parameter family of orientation preserving isometries Φ_t of S^3 such that $\tilde{\rho} \circ \phi_t = \Phi_t \circ \tilde{\rho}$ for all $t \in \mathbb{R}$.*

The next theorem is a consequence of the uniformization theorem applied to a minimal torus in S^3 .

Theorem 2.4. *For every minimal immersion of a torus $\tilde{\rho} : M \rightarrow S^3$, there exists a covering map $\tau : \mathbb{R}^2 \rightarrow M$, a doubly periodic conformal immersion $\rho : \mathbb{R}^2 \rightarrow S^3$, a Gauss map $\nu : \mathbb{R}^2 \rightarrow S^3$, and a fixed angle α , so that*

$$\rho(u, v) = \tilde{\rho}(\tau(u, v)), \quad \nu(u, v) \perp \rho_*(T_{(u,v)}\mathbb{R}^2), \quad \nu(u, v) \perp \rho(u, v),$$

and

$$\begin{aligned} \frac{\partial^2 \rho}{\partial u^2} &= -\frac{\partial r}{\partial u} \frac{\partial \rho}{\partial u} + \frac{\partial r}{\partial v} \frac{\partial \rho}{\partial v} + \cos(2\alpha)\nu - e^{-2r}\rho \\ \frac{\partial^2 \rho}{\partial v^2} &= \frac{\partial r}{\partial u} \frac{\partial \rho}{\partial u} - \frac{\partial r}{\partial v} \frac{\partial \rho}{\partial v} - \cos(2\alpha)\nu - e^{-2r}\rho \\ \frac{\partial^2 \rho}{\partial u \partial v} &= -\frac{\partial r}{\partial v} \frac{\partial \rho}{\partial u} - \frac{\partial r}{\partial u} \frac{\partial \rho}{\partial v} - \sin(2\alpha)\nu \\ \frac{\partial \nu}{\partial u} &= e^{2r}(-\cos(2\alpha)\frac{\partial \rho}{\partial u} + \sin(2\alpha)\frac{\partial \rho}{\partial v}) \\ \frac{\partial \nu}{\partial v} &= e^{2r}(\sin(2\alpha)\frac{\partial \rho}{\partial u} + \cos(2\alpha)\frac{\partial \rho}{\partial v}) \end{aligned}$$

where $e^{-2r} = \langle \frac{\partial \rho}{\partial u}, \frac{\partial \rho}{\partial u} \rangle = \langle \frac{\partial \rho}{\partial v}, \frac{\partial \rho}{\partial v} \rangle$. Moreover, $\Delta r + 2 \sinh(2r) = 0$.

Proof. The idea of the proof is the following: the existence of the conformal map ρ and the covering τ follows from the uniformization theorem, the existence of the constant α follows from the fact that

$$f(z) = f(u + iv) = \langle \frac{\partial^2 \rho}{\partial u^2}, \nu \rangle - i \langle \frac{\partial^2 \rho}{\partial u \partial v}, \nu \rangle$$

is an analytic, doubly periodic function in the whole plane, and therefore it is constant. Clearly this constant function f is not identically zero otherwise M would be totally geodesic. By scaling the coordinates u and v by a constant, we can make $f(u + iv) = \cos(2\alpha) + i \sin(2\alpha)$ for some constant angle α .

To complete the proof, the equations for the second derivatives of ρ are just the standard computation of the Christoffel symbols and the elliptic equation of r follows from computing the Gauss curvature using the Christoffel symbols and making it equal to $1 - e^{4r}$, i.e., this elliptic equation follows from the Gauss equation. \square

Corollary 2.5. *Using the same notation as in the previous theorem, the principal directions of the minimal immersion are given by*

$$V_1 = e^r(\cos(\alpha)\frac{\partial \rho}{\partial u} - \sin(\alpha)\frac{\partial \rho}{\partial v}) \quad \text{and} \quad V_2 = e^r(\sin(\alpha)\frac{\partial \rho}{\partial u} + \cos(\alpha)\frac{\partial \rho}{\partial v}).$$

More precisely,

$$d\nu(\{d\rho_{(u,v)}\}^{-1}(W_1 \circ \tau)) = -e^{2r}V_1 \quad \text{and} \quad d\nu(\{d\rho_{(u,v)}\}^{-1}(W_2 \circ \tau)) = e^{2r}V_2.$$

Moreover, it follows from the last expression that the principal curvatures are $\pm a$ where the function $a : M \rightarrow \mathbb{R}$ satisfies $a(\tau(u, v)) = e^{2r(u,v)}$.

Remark 2.6. A direct computation shows that if make a rotation of the variable u and v , i.e. if we consider the variables \bar{u} and \bar{v} where

$$u = \cos(\beta)\bar{u} + \sin(\beta)\bar{v} \quad \text{and} \quad v = -\sin(\beta)\bar{u} + \cos(\beta)\bar{v},$$

then, the angle α in the theorem above changes from α to $\alpha - \beta$.

Corollary 2.7. *If $M \subset S^3$ is a minimal immersed torus, $N : M \rightarrow S^3$ its Gauss map, and $W_1 : M \rightarrow S^3$ and $W_2 : M \rightarrow S^3$ are unit vector field that define the principal directions with $dN_m(W_1) = -aW_1$ and $dN_m(W_2) = aW_2$, where $a : M \rightarrow \mathbb{R}$ is the positive principal curvature function, then*

$$\begin{aligned} \bar{\nabla}_{W_1} W_1 &= \frac{W_2(a)}{2a} W_2 + aN - m \\ \bar{\nabla}_{W_1} W_2 &= -\frac{W_2(a)}{2a} W_1 = \nabla_{W_1} W_2 \\ \bar{\nabla}_{W_2} W_1 &= -\frac{W_1(a)}{2a} W_2 = \nabla_{W_2} W_1 \\ \bar{\nabla}_{W_2} W_2 &= \frac{W_1(a)}{2a} W_1 - aN - m \end{aligned}$$

where $\bar{\nabla}$ is the Levi-Civita Connection in \mathbb{R}^4 and ∇ is the connection in the surface M .

Proof. This follows from Theorem (2.4) and Corollary (2.5) by noticing that

$$W_1(\tau(u, v)) = V_1(u, v), \quad W_2(\tau(u, v)) = V_2(u, v) \quad \text{and} \quad N(\tau(u, v)) = \nu(u, v).$$

□

Remark 2.8. Theorem (2.2) can be proved using Theorem (2.4) in the following way: Take maps $\rho, V_1, V_2, \nu : \mathbb{R}^2 \rightarrow S^3$, $\tau : \mathbb{R}^2 \rightarrow M$ and $r : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that they satisfy the condition of Theorem (2.4) with $\alpha = 0$, i.e., with $V_1(u, v) = W_1(\tau(u, v)) = e^{r(u, v)} \frac{\partial \rho}{\partial u}(u, v)$ and $V_2(u, v) = W_2(\tau(u, v)) = e^{r(u, v)} \frac{\partial \rho}{\partial v}(u, v)$. Since $\Delta_{\mathbb{R}^2} r + 2 \sinh(2r) = 0$ we obtain that

$$\Delta_{\mathbb{R}^2} \frac{\partial r}{\partial u} + 4 \cosh(2r) \frac{\partial r}{\partial u} = 0.$$

Since $\frac{\partial \rho}{\partial u}(u, v) = e^{-r} V_1(u, v) = e^{-r} W_1(\tau(u, v))$ and $a(\tau(u, v)) = e^{2r(u, v)}$, we have

$$\frac{\partial r}{\partial u} = a^{-\frac{1}{2}} W_1 \left(\frac{1}{2} \ln(a) \right) = \frac{1}{2} a^{-\frac{3}{2}} W_1(a).$$

Denote by Δ_M the Laplacian in the surface. Since the metric induced by ρ in \mathbb{R}^2 is given by $ds^2 = e^{-2r}(du^2 + dv^2)$, we obtain that,

$$\Delta_M \left(\frac{1}{2} a^{-\frac{3}{2}} W_1(a) \right) = a \Delta_{\mathbb{R}^2} \left(\frac{\partial r}{\partial u} \right) = -a(2(a + a^{-1}) \left(\frac{1}{2} a^{-\frac{3}{2}} W_1(a) \right))$$

Therefore the function $h_0 = a^{-\frac{3}{2}} W_1(a)$ satisfies $J(h_0) = 0$. We prove that $J(h_{\frac{\pi}{2}}) = 0$ similarly.

3. INTEGRABLE SYSTEMS AND SOLUTIONS OF THE SINH-GORDON EQUATION

In this section we will study an integrable system that produces solutions of the sinh-Gordon equations, the construction made here is similar to that of [1] and [5]. The integrable system lives in

$$\mathbb{R}^{18} = \{(p, V_1, V_2, \nu, r, s) : p, V_1, V_2, \nu \in \mathbb{R}^4, \quad \text{and} \quad r, s \in \mathbb{R}\}$$

and is given by the vector fields $Z, W : \mathbb{R}^{18} \rightarrow \mathbb{R}^{18}$ given by

$$\begin{aligned} Z &= (e^{-r}(\cos(\theta)V_1 + \sin(\theta)V_2), sV_2 + \cos(\theta)(e^r\nu - e^{-r}p), \\ &\quad -sV_1 - \sin(\theta)(e^r\nu + e^{-r}p), e^r(-\cos(\theta)V_1 + \sin(\theta)V_2), \\ &\quad \langle Bp, \nu \rangle, \cos(\theta)\langle BV_2, e^{-r}\nu - e^r p \rangle - \sin(\theta)\langle BV_1, e^{-r}\nu + e^r p \rangle) \\ &= (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6) \\ \\ W &= (e^{-r}(\cos(\theta)V_2 - \sin(\theta)V_1), -\langle Bp, \nu \rangle V_2 + \sin(\theta)(-e^r\nu + e^{-r}p), \\ &\quad \langle Bp, \nu \rangle V_1 - \cos(\theta)(e^r\nu + e^{-r}p), e^r(\sin(\theta)V_1 + \cos(\theta)V_2), s, \\ &\quad e^{-2r} - e^{2r} - \sin(\theta)\langle BV_2, e^{-r}\nu - e^r p \rangle - \cos(\theta)\langle BV_1, e^{-r}\nu + e^r p \rangle) \\ &= (W_1, W_2, W_3, W_4, W_5, W_6), \end{aligned}$$

where B is a skew-symmetric 4×4 matrix, i.e. $B \in so(4)$, and θ is any real number. In the notation above, W_i and Z_i have values in \mathbb{R}^4 for $i = 1, 2, 3, 4$, and in \mathbb{R} for $i = 5, 6$. We can write the system induced by these vector fields as

$$\begin{aligned} \frac{\partial p}{\partial u} &= e^{-r}(\cos(\theta)V_1 + \sin(\theta)V_2) \\ \frac{\partial V_1}{\partial u} &= sV_2 + \cos(\theta)(e^r\nu - e^{-r}p) \\ \frac{\partial V_2}{\partial u} &= -sV_1 - \sin(\theta)(e^r\nu + e^{-r}p) \\ \frac{\partial \nu}{\partial u} &= e^r(-\cos(\theta)V_1 + \sin(\theta)V_2) \\ \frac{\partial r}{\partial u} &= \langle Bp, \nu \rangle \\ \frac{\partial s}{\partial u} &= \cos(\theta)\langle BV_2, e^{-r}\nu - e^r p \rangle - \sin(\theta)\langle BV_1, e^{-r}\nu + e^r p \rangle \end{aligned}$$

and

$$\begin{aligned} \frac{\partial p}{\partial v} &= e^{-r}(\cos(\theta)V_2 - \sin(\theta)V_1) \\ \frac{\partial V_1}{\partial v} &= -\langle Bp, \nu \rangle V_2 + \sin(\theta)(-e^r\nu + e^{-r}p) \\ (1) \quad \frac{\partial V_2}{\partial v} &= \langle Bp, \nu \rangle V_1 - \cos(\theta)(e^r\nu + e^{-r}p) \\ \frac{\partial \nu}{\partial v} &= e^r(\sin(\theta)V_1 + \cos(\theta)V_2) \\ \frac{\partial r}{\partial v} &= s \\ \frac{\partial s}{\partial v} &= e^{-2r} - e^{2r} - \sin(\theta)\langle BV_2, e^{-r}\nu - e^r p \rangle - \cos(\theta)\langle BV_1, e^{-r}\nu + e^r p \rangle \end{aligned}$$

We will refer to the previous system as the *integrable system* (1).

The following theorem provides a family a solutions of the sinh-Gordon equation.

Theorem 3.1. *The vector fields Z and W commute, and if*

$$\Theta_Z : (-\epsilon, \epsilon) \times \mathbb{R}^{18} \rightarrow \mathbb{R}^{18} \quad \text{and} \quad \Theta_W : (-\epsilon, \epsilon) \times \mathbb{R}^{18} \rightarrow \mathbb{R}^{18}$$

are the flows of the vector fields Z and W respectively, and for any $\mathbf{x}_0 \in \mathbb{R}^{18}$, we define the map $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^{18}$ by

$$\phi(u, v) = \Theta_Z(u, \Theta_W(v, \mathbf{x}_0)) = (\phi_1(u, v), \dots, \phi_{18}(u, v)),$$

then, the function $r(u, v) = \phi_{17}(u, v)$ solves the equation

$$\Delta r + 2 \sinh(2r) = 0.$$

Proof. We will denote by DW and DZ the 18×18 matrices of the first derivatives of W and Z respectively. We will also denote by $V(f)$ the directional derivative of the function f in the direction of V , the function f may be a vector value function. For example

$$Z_5(f) = \langle Bp, \nu \rangle \frac{\partial}{\partial r}(f) \quad \text{and} \quad W_1(f) = \sum_{i=1}^4 e^{-r} (\cos(\theta) V_2^i - \sin(\theta) V_2^i) \frac{\partial}{\partial p^i}(f)$$

Notice that

$$Z_1(p) = Z_1, \quad Z_1(V_i) = 0, \quad Z_1(\nu) = 0, \quad Z_1(e^{\pm r}) = 0, \quad Z_1(s) = 0$$

We get similar equations for $Z_i(p)$, $Z_i(V_j)$, $Z_i(\nu)$, $Z_i(e^{\pm r})$, $Z_i(s)$, $W_i(p)$, $W_i(V_j)$, $W_i(\nu)$, $W_i(e^{\pm r})$, and $W_i(s)$.

Now,

$$\begin{aligned} [Z, W] &= (DW)Z - (DZ)W = ZW - WZ \\ &= (Z(W_1), Z(W_2), Z(W_3), Z(W_4), Z(W_5), Z(W_6)) - \\ &\quad (W(Z_1), W(Z_2), W(Z_3), W(Z_4), W(Z_5), W(Z_6)) \end{aligned}$$

The first four components of the vector above are given by

$$\begin{aligned} Z(W_1) - W(Z_1) &= e^{-r} \cos(\theta) Z_3 - e^{-r} \sin(\theta) Z_2 - e^{-r} (\cos(\theta) V_2 - \sin(\theta) V_1) Z_5 \\ &\quad - (e^{-r} \cos(\theta) W_2 + e^{-r} \sin(\theta) W_3 - e^{-r} (\cos(\theta) V_1 + \sin(\theta) V_2) W_5) \\ &= e^{-r} \{ \cos(\theta) (-s V_1 - \sin(\theta) (e^r \nu + e^{-r} p)) - \sin(\theta) (s V_2 \\ &\quad + \cos(\theta) (-e^{-r} p + e^r \nu)) \} - \langle Bp, \nu \rangle e^{-r} (\cos(\theta) V_2 - \sin(\theta) V_1) \\ &\quad - e^{-r} (\cos(\theta) (-\langle Bp, \nu \rangle V_2 + \sin(\theta) (e^{-r} p - e^r \nu)) + \sin(\theta) (\langle Bp, \nu \rangle V_1 \\ &\quad - \cos(\theta) (e^{-r} p + e^r \nu))) + e^{-r} (\cos(\theta) V_1 + \sin(\theta) V_2) s \\ &= 0. \end{aligned}$$

Direct computations, some of them longer, some others shorter than the above, show that the other components of $[Z, W]$ are also zero. We now show that $r(u, v) = \phi_{17}(u, v)$ is a solution of the sinh-Gordon equation. We have that

$$\begin{aligned} \Delta r &= \frac{\partial^2 r}{\partial u^2} + \frac{\partial^2 r}{\partial v^2} = \frac{\partial \langle Bp, \nu \rangle}{\partial u} + \frac{\partial s}{\partial v} \\ &= \langle B(e^{-r} (\cos(\theta) V_1 + \sin(\theta) V_2)), \nu \rangle + \langle Bp, e^r (-\cos(\theta) V_1 + \sin(\theta) V_2) \rangle \\ &\quad - 2 \sinh(2r) - \sin(\theta) \langle BV_2, e^{-r} \nu - e^r p \rangle - \cos(\theta) \langle BV_1, e^{-r} \nu + e^r p \rangle \\ &= -2 \sinh(2r). \end{aligned}$$

Notice that in the last step we have used the fact that $\langle Bp, V_2 \rangle = -\langle BV_2, p \rangle$, i.e. we have used the fact that $B^T = -B$. \square

The previous theorem shows that for any choice of $B \in so(4)$, $\theta \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^{18}$ we have a solution of the sinh-Gordon equation. One may think that a similar integrable system in $\mathbb{R}^{4n+2} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^2$ can be defined so that it produces a bigger space of solutions for the sinh-Gordon equation. Indeed the system (1) can be generalized to an integrable system in \mathbb{R}^{4n+2} for any given $B \in so(n)$ and $\theta \in \mathbb{R}$, but the solutions of the sinh-Gordon will reduce to solutions in the case $n = 1$, as the following proposition explains.

Proposition 3.2. *If we think about the integrable system (1) as being defined in \mathbb{R}^{4n+2} by taking the vectors p, V_1, V_2 and ν in \mathbb{R}^n instead of vectors in \mathbb{R}^4 , and by taking a skew symmetry matrix $B \in so(n)$ instead of a matrix in $so(4)$, then*

- (a) *The new system is integrable.*
- (b) *If $\phi : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{4n+2}$ is the solution of the new system with initial condition $x_0 = (p^0, V_1^0, V_2^0, \nu^0, (r^0, s^0)) \in \mathbb{R}^{4n+2}$, then*

$$\begin{aligned} \phi(u, v) = & (\tilde{\phi}_1 \tilde{e}_1 + \tilde{\phi}_2 \tilde{e}_2 + \tilde{\phi}_3 \tilde{e}_3 + \tilde{\phi}_4 \tilde{e}_4, \tilde{\phi}_5 \tilde{e}_1 + \tilde{\phi}_6 \tilde{e}_2 + \tilde{\phi}_7 \tilde{e}_3 + \tilde{\phi}_8 \tilde{e}_4, \tilde{\phi}_9 \tilde{e}_1 + \tilde{\phi}_{10} \tilde{e}_2 + \\ & \tilde{\phi}_{11} \tilde{e}_3 + \tilde{\phi}_{12} \tilde{e}_4, \tilde{\phi}_{13} \tilde{e}_1 + \tilde{\phi}_{14} \tilde{e}_2 + \tilde{\phi}_{15} \tilde{e}_3 + \tilde{\phi}_{16} \tilde{e}_4, (\tilde{\phi}_{17}, \tilde{\phi}_{18})) \end{aligned}$$

where $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ is an orthonormal basis of any 4 dimensional subspace S of \mathbb{R}^n that contains the vectors p^0, V_1^0, V_2^0 and ν^0 , and $\tilde{\phi} : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^{18}$ is the solution of the integrable system (1) with initial condition $(\tilde{p}^0, \tilde{V}_1^0, \tilde{V}_2^0, \tilde{\nu}^0, (r^0, s^0))$, where $\tilde{p}^0, \tilde{V}_1^0, \tilde{V}_2^0, \tilde{\nu}^0$ are the coordinates of the vectors p^0, V_1^0, V_2^0, ν^0 in S with respect to the basis $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$, and matrix $\tilde{B} \in so(4)$ given by $\tilde{b}_{ij} = \langle \tilde{B} \tilde{e}_i, \tilde{e}_j \rangle$.

Proof. The proof of part (a) follows the same lines as the first part in the proof of Theorem 3.1. Part (b) follows from the fact that, if for any $y \in \mathbb{R}^4$ we define $\hat{y} = y_1 \tilde{e}_1 + y_2 \tilde{e}_2 + y_3 \tilde{e}_3 + y_4 \tilde{e}_4$, then $\langle B \hat{y}, \hat{z} \rangle = \langle \tilde{B} y, z \rangle$ and $\widehat{F'(u)} = \hat{F}'(u)$ for any differentiable map $F : \mathbb{R} \rightarrow \mathbb{R}^4$. \square

In order to study the system (1) we will define this second integrable system,

Theorem 3.3. *Let $p, \nu, V_1, V_2 : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^4$ and $r, s : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ be a solution of the integrable system (1). If \tilde{B} is another skew symmetric matrix, and if we define the functions*

$$\begin{aligned} \xi_1 &= \langle Bp, \nu \rangle, \xi_2 = \langle BV_1, p \rangle, \xi_3 = \langle BV_2, p \rangle, \xi_4 = \langle BV_1, V_2 \rangle, \xi_5 = \langle BV_1, \nu \rangle, \xi_6 = \langle BV_2, \nu \rangle \\ \tilde{\xi}_1 &= \langle \tilde{B}p, \nu \rangle, \tilde{\xi}_2 = \langle \tilde{B}V_1, p \rangle, \tilde{\xi}_3 = \langle \tilde{B}V_2, p \rangle, \tilde{\xi}_4 = \langle \tilde{B}V_1, V_2 \rangle, \tilde{\xi}_5 = \langle \tilde{B}V_1, \nu \rangle, \tilde{\xi}_6 = \langle \tilde{B}V_2, \nu \rangle, \end{aligned}$$

then

$$\begin{aligned}
\frac{\partial \xi_1}{\partial u} &= e^r (\cos(\theta) \xi_2 - \sin(\theta) \xi_3) + e^{-r} (\cos(\theta) \xi_5 + \sin(\theta) \xi_6) \\
\frac{\partial \xi_2}{\partial u} &= s \xi_3 - e^r \cos(\theta) \xi_1 + e^{-r} \sin(\theta) \xi_4 \\
\frac{\partial \xi_3}{\partial u} &= -s \xi_2 + e^r \sin(\theta) \xi_1 - e^{-r} \cos(\theta) \xi_4 \\
\frac{\partial \xi_4}{\partial u} &= e^r (-\sin(\theta) \xi_5 - \cos(\theta) \xi_6) + e^{-r} (\cos(\theta) \xi_3 - \sin(\theta) \xi_2) \\
\frac{\partial \xi_5}{\partial u} &= s \xi_6 + e^r \sin(\theta) \xi_4 - e^{-r} \cos(\theta) \xi_1 \\
\frac{\partial \xi_6}{\partial u} &= -s \xi_5 + e^r \cos(\theta) \xi_4 - e^{-r} \sin(\theta) \xi_1 \\
\frac{\partial r}{\partial u} &= \xi_1 \\
\frac{\partial s}{\partial u} &= e^r (-\cos(\theta) \xi_3 - \sin(\theta) \xi_2) + e^{-r} (\cos(\theta) \xi_6 - \sin(\theta) \xi_5) \\
\\
\frac{\partial \tilde{\xi}_1}{\partial u} &= e^r (\cos(\theta) \tilde{\xi}_2 - \sin(\theta) \tilde{\xi}_3) + e^{-r} (\cos(\theta) \tilde{\xi}_5 + \sin(\theta) \tilde{\xi}_6) \\
\frac{\partial \tilde{\xi}_2}{\partial u} &= s \tilde{\xi}_3 - e^r \cos(\theta) \tilde{\xi}_1 + e^{-r} \sin(\theta) \tilde{\xi}_4 \\
\frac{\partial \tilde{\xi}_3}{\partial u} &= -s \tilde{\xi}_2 + e^r \sin(\theta) \tilde{\xi}_1 - e^{-r} \cos(\theta) \tilde{\xi}_4 \\
\frac{\partial \tilde{\xi}_4}{\partial u} &= e^r (-\sin(\theta) \tilde{\xi}_5 - \cos(\theta) \tilde{\xi}_6) + e^{-r} (\cos(\theta) \tilde{\xi}_3 - \sin(\theta) \tilde{\xi}_2) \\
\frac{\partial \tilde{\xi}_5}{\partial u} &= s \tilde{\xi}_6 + e^r \sin(\theta) \tilde{\xi}_4 - e^{-r} \cos(\theta) \tilde{\xi}_1 \\
\frac{\partial \tilde{\xi}_6}{\partial u} &= -s \tilde{\xi}_5 + e^r \cos(\theta) \tilde{\xi}_4 - e^{-r} \sin(\theta) \tilde{\xi}_1
\end{aligned}$$

and,

$$\begin{aligned}
\frac{\partial \xi_1}{\partial v} &= -e^r (\cos(\theta) \xi_3 + \sin(\theta) \xi_2) + e^{-r} (\cos(\theta) \xi_6 - \sin(\theta) \xi_5) \\
\frac{\partial \xi_2}{\partial v} &= -\xi_1 \xi_3 + e^r \sin(\theta) \xi_1 + e^{-r} \cos(\theta) \xi_4 \\
\frac{\partial \xi_3}{\partial v} &= \xi_1 \xi_2 + e^r \cos(\theta) \xi_1 + e^{-r} \sin(\theta) \xi_4 \\
\frac{\partial \xi_4}{\partial v} &= e^r (\sin(\theta) \xi_6 - \cos(\theta) \xi_5) + e^{-r} (-\cos(\theta) \xi_2 - \sin(\theta) \xi_3) \\
\frac{\partial \xi_5}{\partial v} &= -\xi_1 \xi_6 + e^r \cos(\theta) \xi_4 + e^{-r} \sin(\theta) \xi_1 \\
\frac{\partial \xi_6}{\partial v} &= \xi_1 \xi_5 - e^r \sin(\theta) \xi_4 - e^{-r} \cos(\theta) \xi_1 \\
\frac{\partial r}{\partial v} &= s \\
\frac{\partial s}{\partial v} &= -2 \sinh(2r) + e^r (\sin(\theta) \xi_3 - \cos(\theta) \xi_2) + e^{-r} (-\sin(\theta) \xi_6 - \cos(\theta) \xi_5)
\end{aligned}
\tag{2}$$

$$\begin{aligned}
\frac{\partial \tilde{\xi}_1}{\partial v} &= -e^r(\cos(\theta)\tilde{\xi}_3 + \sin(\theta)\tilde{\xi}_2) + e^{-r}(\cos(\theta)\tilde{\xi}_6 - \sin(\theta)\tilde{\xi}_5) \\
\frac{\partial \tilde{\xi}_2}{\partial v} &= -\xi_1\tilde{\xi}_3 + e^r \sin(\theta)\tilde{\xi}_1 + e^{-r} \cos(\theta)\tilde{\xi}_4 \\
\frac{\partial \tilde{\xi}_3}{\partial v} &= \xi_1\tilde{\xi}_2 + e^r \cos(\theta)\tilde{\xi}_1 + e^{-r} \sin(\theta)\tilde{\xi}_4 \\
\frac{\partial \tilde{\xi}_4}{\partial v} &= e^r(\sin(\theta)\tilde{\xi}_6 - \cos(\theta)\tilde{\xi}_5) + e^{-r}(-\cos(\theta)\tilde{\xi}_2 - \sin(\theta)\tilde{\xi}_3) \\
\frac{\partial \tilde{\xi}_5}{\partial v} &= -\xi_1\tilde{\xi}_6 + e^r \cos(\theta)\tilde{\xi}_4 + e^{-r} \sin(\theta)\tilde{\xi}_1 \\
\frac{\partial \tilde{\xi}_6}{\partial v} &= \xi_1\tilde{\xi}_5 - e^r \sin(\theta)\tilde{\xi}_4 - e^{-r} \cos(\theta)\tilde{\xi}_1
\end{aligned}$$

Moreover, The system given by the equations (2) is integrable.

Proof. This is long direct computation. □

3.1. First integrals and existence of global solutions. In this subsection we will prove that the solutions of the sinh-Gordon equations given by the integrable system (1) are defined in the whole of \mathbb{R}^2 . In order to prove this, we first establish some lemmas.

Lemma 3.4. *For a given solution of the system (1), the functions ξ_1, \dots, ξ_6 defined in Theorem (3.3) satisfy*

$$M = \frac{1}{2}\{\xi_1^2 + \dots + \xi_6^2\}$$

is a constant.

Proof. A direct computation using Theorem (3.3) gives us that

$$\begin{aligned}
\frac{\partial M}{\partial u} &= \xi_1 \frac{\partial \xi_1}{\partial u} + \dots + \xi_6 \frac{\partial \xi_6}{\partial u} \\
&= \xi_1(e^r(\cos(\theta)\xi_2 - \sin(\theta)\xi_3) + e^{-r}(\cos(\theta)\xi_5 + \sin(\theta)\xi_6)) \\
&\quad + \xi_2(s\xi_3 - e^r \cos(\theta)\xi_1 + e^{-r} \sin(\theta)\xi_4) \\
&\quad + \xi_3(-s\xi_2 + e^r \sin(\theta)\xi_1 - e^{-r} \cos(\theta)\xi_4) \\
&\quad + \xi_4(e^r(-\sin(\theta)\xi_5 - \cos(\theta)\xi_6) + e^{-r}(\cos(\theta)\xi_3 - \sin(\theta)\xi_2)) \\
&\quad + \xi_5(s\xi_6 + e^r \sin(\theta)\xi_4 - e^{-r} \cos(\theta)\xi_1) \\
&\quad + \xi_6(-s\xi_5 + e^r \cos(\theta)\xi_4 - e^{-r} \sin(\theta)\xi_1) \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial M}{\partial v} &= \xi_1 \frac{\partial \xi_1}{\partial v} + \cdots + \xi_6 \frac{\partial \xi_6}{\partial v} \\
&= \xi_1(-e^r(\cos(\theta)\xi_3 + \sin(\theta)\xi_2) + e^{-r}(\cos(\theta)\xi_6 - \sin(\theta)\xi_5)) \\
&\quad + \xi_2(-\xi_1\xi_3 + e^r \sin(\theta)\xi_1 + e^{-r} \cos(\theta)\xi_4) \\
&\quad + \xi_3(\xi_1\xi_2 + e^r \cos(\theta)\xi_1 + e^{-r} \sin(\theta)\xi_4) \\
&\quad + \xi_4(e^{-r}(-\cos(\theta)\xi_2 - \sin(\theta)\xi_3)) \\
&\quad + \xi_5(-\xi_1\xi_6 + e^r \cos(\theta)\xi_4 + e^{-r} \sin(\theta)\xi_1) \\
&\quad + \xi_6(\xi_1\xi_5 - e^r \sin(\theta)\xi_4 - e^{-r} \cos(\theta)\xi_1) \\
&= 0,
\end{aligned}$$

therefore, M is a constant. \square

Lemma 3.5. *For a given solution of the system (1), the functions p, V_1, V_2, ν satisfy*

$$E = \frac{1}{2}\{\langle p, p \rangle + \langle V_1, V_1 \rangle + \langle V_2, V_2 \rangle + \langle \nu, \nu \rangle\}$$

is a constant.

Proof. As in the proof of the previous lemma, a direct computation shows that $\frac{\partial E}{\partial u} = \frac{\partial E}{\partial v} = 0$. \square

Lemma 3.6. *For a given solution of the system (1), the functions ξ_1, \dots, ξ_6 defined in Theorem (3.3) satisfy*

$$A = e^r(\cos(\theta)\xi_2 - \sin(\theta)\xi_3) - e^{-r}(\cos(\theta)\xi_5 + \sin(\theta)\xi_6) + \frac{1}{2}s^2 + \cosh(2r) - \frac{1}{2}(\xi_1)^2$$

is a constant.

Proof. Similarly to the previous two lemmas, we prove that $\frac{\partial A}{\partial u} = \frac{\partial A}{\partial v} = 0$.

Denote by

$$\begin{aligned}
B &= e^r(\cos(\theta)\xi_2 - \sin(\theta)\xi_3) - e^{-r}(\cos(\theta)\xi_5 + \sin(\theta)\xi_6) \text{ and} \\
C &= \frac{\partial \xi_1}{\partial u} = e^r(\cos(\theta)\xi_2 - \sin(\theta)\xi_3) + e^{-r}(\cos(\theta)\xi_5 + \sin(\theta)\xi_6).
\end{aligned}$$

Notice that $B + \frac{1}{2}s^2 - \frac{1}{2}\xi_1^2 + \cosh(2r) = A$. A direct computation shows that

$$\begin{aligned}
\frac{\partial B}{\partial u} &= \xi_1 C + e^r\{\cos(\theta)(s\xi_3 - e^r \cos(\theta)\xi_1 + e^{-r} \sin(\theta)\xi_4) \\
&\quad - \sin(\theta)(-s\xi_2 + e^r \sin(\theta)\xi_1 - e^{-r} \cos(\theta)\xi_4)\} \\
&\quad - e^{-r}\{\cos(\theta)(s\xi_6 + e^r \sin(\theta)\xi_4 - e^{-r} \cos(\theta)\xi_1) \\
&\quad + \sin(\theta)(-s\xi_5 + e^r \cos(\theta)\xi_4 - e^{-r} \sin(\theta)\xi_1)\} \\
&= \xi_1 \frac{\partial \xi_1}{\partial u} + s(e^r \cos(\theta)\xi_3 + e^r \sin(\theta)\xi_2 - e^{-r} \cos(\theta)\xi_6 + e^{-r} \sin(\theta)\xi_5) \\
&\quad + \xi_4(\cos(\theta) \sin(\theta) + \cos(\theta) \sin(\theta) - \cos(\theta) \sin(\theta) - \cos(\theta) \sin(\theta)) \\
&\quad + \xi_1(-e^{2r} \cos^2(\theta)\xi_3 - e^{2r} \sin^2(\theta)\xi_2 + e^{-2r} \cos^2(\theta) + e^{-2r} \sin^2(\theta)) \\
&= \xi_1 \frac{\partial \xi_1}{\partial u} - s \frac{\partial s}{\partial u} - 2\xi_1 \sinh(2r) \\
&= \frac{1}{2} \frac{\partial \xi_1^2}{\partial u} - \frac{1}{2} \frac{\partial s^2}{\partial u} - \frac{\partial \cosh(2r)}{\partial u}.
\end{aligned}$$

Therefore $\frac{\partial A}{\partial u} = 0$. Similarly,

$$\begin{aligned}
\frac{\partial B}{\partial v} &= sC + e^r \{ \cos(\theta)(-\xi_1 \xi_3 + e^r \sin(\theta) \xi_1 + e^{-r} \cos(\theta) \xi_4) \\
&\quad - \sin(\theta)(-\xi_1 \xi_2 + e^r \cos(\theta) \xi_1 + e^{-r} \sin(\theta) \xi_4) \} \\
&\quad - e^{-r} \{ \cos(\theta)(-\xi_1 \xi_6 + e^r \cos(\theta) \xi_4 + e^{-r} \sin(\theta) \xi_1) \\
&\quad + \sin(\theta)(\xi_1 \xi_5 - e^r \sin(\theta) \xi_4 - e^{-r} \cos(\theta) \xi_1) \} \\
&= s(-2 \sinh(2r) - \frac{\partial s}{\partial v}) + \xi_1(-e^r \cos(\theta) \xi_3 + e^{2r} \cos(\theta) \sin(\theta) \\
&\quad - e^{2r} \sin(\theta) \cos(\theta) - e^r \sin(\theta) \xi_2) \\
&\quad + e^{-r} \cos(\theta) \xi_6 - e^{-2r} \cos(\theta) \sin(\theta) - e^{-r} \sin(\theta) \xi_5 + e^{-2r} \sin(\theta) \cos(\theta) \\
&\quad + \xi_4(\cos^2(\theta) - \sin^2(\theta) + \cos^2(\theta) + \sin^2(\theta)) \\
&= -\frac{1}{2} \frac{\partial s^2}{\partial v} - \frac{\partial \cosh(2r)}{\partial v} + \frac{1}{2} \frac{\partial \xi_1^2}{\partial v}.
\end{aligned}$$

□

Corollary 3.7. *Given a solution of the system (1). If M and A are the constants given by Lemmas (3.4) and (3.6), respectively, if (u_0, v_0) is any point in the domain of the solution, and if R is a real number such that*

$$\cosh(2R) > A + 4M \cosh(R) + \frac{M^2}{2} \quad \text{and} \quad R > |r(u_0, v_0)|$$

Then, $|r(u, v)| < R$ and

$$\frac{1}{2} s^2(u, v) + \cosh(2r(u, v)) \leq A + \frac{M^2}{2} + \cosh(2R)$$

for any (u, v) in the domain of the solution.

Proof. We have that

$$\begin{aligned}
\frac{1}{2} s^2(u, v) + \cosh(2r(u, v)) &= A + \frac{1}{2} \xi_1^2 + e^{-r} (\cos(\theta) \xi_5 + \sin(\theta) \xi_6) - e^r (\cos(\theta) \xi_2 - \sin(\theta) \xi_3) \\
&\leq A + \frac{M^2}{2} + 4M \cosh(r).
\end{aligned}$$

This inequality above shows that the result will follow once we prove that $|r(u, v)| \leq R$. We prove that $|r(u, v)| < R$ by contradiction. If, for some (u, v) , $|r(u, v)| = R$, then, the inequality above implies that at that (u, v) ,

$$\cosh(2R) \leq A + \frac{M^2}{2} + 4M \cosh(R).$$

This is a contradiction with the choice of R given in the hypotheses. Therefore the Corollary follows. □

Theorem 3.8. *Any solution of the system (1) is defined on the entirety of \mathbb{R}^2 .*

Proof. By Lemma (3.4), Lemma (3.5), and Corollary (3.7), the solution of the system (1) remains bounded in \mathbb{R}^{18} for all (u, v) , guaranteeing the existence of the solution for all (u, v) . □

3.2. The integrable system and minimal immersion of the plane. In this section we prove that if we choose \mathbf{x}_0 properly, a solution of the system (1) produces an example of a minimal immersion of \mathbb{R}^2 into S^3 . We start by showing that the vector fields Z and W that define the system (1) define vector fields in the manifold $SO(4) \times \mathbb{R}^2$. To see this, consider the set $SO(4) \times \mathbb{R}^2$ as the following subset of \mathbb{R}^{18} :

$$SO(4) \times \mathbb{R}^2 = \{(V_0, V_1, V_2, V_3, (r, s)) \in (\mathbb{R}^4)^4 \times \mathbb{R}^2 : \langle V_i, V_j \rangle = \delta_{ij}, \det[V_0 | \cdots | V_3] = 1\}.$$

With this in mind, it is not difficult to verify the following lemma.

Lemma 3.9. *If a vector field $Y : \mathbb{R}^{18} \rightarrow \mathbb{R}^{18}$ can be written as*

$$(3) \quad \begin{aligned} Y(x) &= (c_{12}(x)V_1 + c_{13}(x)V_2 + c_{14}(x)V_3, -c_{12}(x)V_0 + c_{23}(x)V_2 + c_{24}(x)V_4, \\ &\quad -c_{13}(x)V_0 - c_{23}(x)V_1 + c_{34}(x)V_3, -c_{14}(x)V_0 - c_{24}(x)V_1 - c_{34}(x)V_2, \\ &\quad f(x), g(x)) \\ &= (C(x)[V_0^T, V_1^T, V_2^T, V_3^T], f(x), g(x)), \end{aligned}$$

where $x = (V_0, V_1, V_2, V_3, (r, s))$ denotes a point in \mathbb{R}^{18} , and $C : \mathbb{R}^{18} \rightarrow so(4)$, $f : \mathbb{R}^{18} \rightarrow \mathbb{R}$ and $g : \mathbb{R}^{18} \rightarrow \mathbb{R}$ are smooth functions, then the restriction of Y to $SO(4) \times \mathbb{R}^2$ defines a vector field tangent to $SO(4) \times \mathbb{R}^2$.

As a consequence of this lemma we have:

Lemma 3.10. *The vector fields Z and W that define the integrable system (1) define tangent vector fields in $SO(4) \times \mathbb{R}^2$.*

Proof. Notice that the notation used in Lemma (3.9) and the notation used in the system (1) are the same after identifying $p = V_0$ and $\nu = V_3$. The lemma follows because the vector field Z can be written in the form of Lemma (3.9) with

$$C_1(x) = \begin{pmatrix} 0 & e^{-r} \cos(\theta) & e^{-r} \sin(\theta) & 0 \\ -e^{-r} \cos(\theta) & 0 & s & e^r \cos(\theta) \\ -e^{-r} \sin(\theta) & -s & 0 & -e^r \sin(\theta) \\ 0 & -e^r \cos(\theta) & e^r \sin(\theta) & 0 \end{pmatrix}$$

and

$$\begin{aligned} f_1(p, V_1, V_2, \nu, (r, s)) &= \langle Bp, \nu \rangle \quad \text{and} \\ g_1(p, V_1, V_2, \nu, (r, s)) &= \cos(\theta) \langle BV_2, e^{-r} \nu - e^r p \rangle - \sin(\theta) \langle BV_1, e^{-r} \nu + e^r p \rangle. \end{aligned}$$

Also, the vector field W can be written in the form of the lemma (3.9) with

$$C_2(x) = \begin{pmatrix} 0 & -e^{-r} \sin(\theta) & e^{-r} \cos(\theta) & 0 \\ e^{-r} \sin(\theta) & 0 & -\langle Bp, \nu \rangle & -e^r \sin(\theta) \\ -e^{-r} \cos(\theta) & \langle Bp, \nu \rangle & 0 & e^r \cos(\theta) \\ 0 & 0 & e^r \sin(\theta) & e^r \cos(\theta) \\ 0 & e^r \sin(\theta) & e^r \cos(\theta) & 0 \end{pmatrix}$$

and

$$\begin{aligned} f_2(p, V_1, V_2, \nu, (r, s)) &= s \quad \text{and} \\ g_2(p, V_1, V_2, \nu, (r, s)) &= -2 \sinh(2r) - \sin(\theta) \langle BV_2, e^{-r} \nu - e^r p \rangle - \cos(\theta) \langle BV_1, e^{-r} \nu + e^r p \rangle. \end{aligned}$$

Lemma (3.9) then implies that Z and W are tangent to $SO(4) \times \mathbb{R}^2$. \square

Theorem 3.11. *Let, Z and W be the vector fields that defined the integrable system (1) and let $\mathbf{x}^0 = (p^0, V_1^0, V_1^0, \nu^0, r^0, s^0) \in \mathbb{R}^{18}$ be such that $\{p^0, V_1^0, V_1^0, \nu^0\}$ is an orthonormal basis of \mathbb{R}^4 and let*

$$\Theta_Z : \mathbb{R} \times \mathbb{R}^{18} \rightarrow \mathbb{R}^{18} \quad \text{and} \quad \Theta_W : \mathbb{R} \times \mathbb{R}^{18} \rightarrow \mathbb{R}^{18}$$

be the flows of the vector fields Z and W respectively. If $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^{18}$ is given by

$$\phi(u, v) = \Theta_Z(u, \Theta_W(v, \mathbf{x}^0)) = (\phi_1(u, v), \dots, \phi_{18}(u, v))$$

then the map

$$\rho : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \text{ given by } \rho(u, v) = (\phi_1(u, v), \dots, \phi_4(u, v))$$

satisfies $M = \rho(\mathbb{R}^2) \subset S^3$, ρ is a minimal immersion of S^3 with principal curvature at $\rho(u, v)$ given by $\pm e^{2\phi_{17}(u, v)}$. More precisely, the map

$$\nu : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \text{ given by } \nu(u, v) = (\phi_{13}(u, v), \phi_{14}(u, v), \phi_{15}(u, v), \phi_{16}(u, v))$$

is the Gauss map of the immersion ρ , and

$$(\phi_5(u, v), \phi_6(u, v), \phi_7(u, v), \phi_8(u, v)) \text{ and } (\phi_9(u, v), \phi_{10}(u, v), \phi_{11}(u, v), \phi_{12}(u, v))$$

are the principal directions of the immersion ρ .

Proof. Denote this solution by $\phi = (p, V_1, V_2, \nu, r, s)$, where $p = \rho, V_1, V_2, \nu : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ and $r, s : \mathbb{R}^2 \rightarrow \mathbb{R}$. By the previous lemma, $|p| = |\rho| = 1$, because the initial conditions belong to $SO(4) \times \mathbb{R}^2$ and therefore the whole solution stays in $SO(4) \times \mathbb{R}^2$. By the form of the vector field Z and W or, equivalently, by the fact that ϕ is a solution of the system (1),

$$\frac{\partial p}{\partial u} = e^{-r}(\cos(\theta)V_1 + \sin(\theta)V_2) \quad \text{and} \quad \frac{\partial p}{\partial v} = e^{-r}(\cos(\theta)V_2 - \sin(\theta)V_1).$$

Again by the fact that the solution remains in $SO(4) \times \mathbb{R}^2$, the first fundamental form of the parameterized surface $\rho = p : \mathbb{R}^2 \rightarrow S^3$ is given by

$$E = \left\langle \frac{\partial p}{\partial u}, \frac{\partial p}{\partial u} \right\rangle = e^{-2r}, \quad F = \left\langle \frac{\partial p}{\partial u}, \frac{\partial p}{\partial v} \right\rangle = 0, \quad \text{and} \quad G = \left\langle \frac{\partial p}{\partial v}, \frac{\partial p}{\partial v} \right\rangle = e^{-2r}.$$

Therefore the immersion ρ is a conformal immersion and for every $(u, v) \in \mathbb{R}^2$, the vectors $\{V_1(u, v), V_2(u, v)\}$ form a basis of the tangent space $T_{\rho(u, v)}M$. More precisely,

$$V_1 = e^r(\cos(\theta)\frac{\partial p}{\partial u} - \sin(\theta)\frac{\partial p}{\partial v}) \quad \text{and} \quad V_2 = e^r(\sin(\theta)\frac{\partial p}{\partial u} + \cos(\theta)\frac{\partial p}{\partial v}).$$

Once again from the fact that the solution remains in $SO(4) \times \mathbb{R}^2$, the map $\nu : \mathbb{R}^2 \rightarrow S^3$ defines the Gauss map of the immersion ρ . Since ϕ is a solution of the system (1),

$$\frac{\partial \nu}{\partial u} = d\nu\left(\frac{\partial p}{\partial u}\right) = e^r(-\cos(\theta)V_1 + \sin(\theta)V_2) \quad \text{and} \quad \frac{\partial \nu}{\partial v} = d\nu\left(\frac{\partial p}{\partial v}\right) = e^r(\sin(\theta)V_1 + \cos(\theta)V_2).$$

In the previous equalities we identify the Gauss map defined in \mathbb{R}^2 with the Gauss map defined in $M = \rho(\mathbb{R}^2) \subset S^3$. The previous equation implies that

$$\begin{aligned} d\nu(V_1) &= d\nu\left(e^r\left(\cos(\theta)\frac{\partial p}{\partial u} - \sin(\theta)\frac{\partial p}{\partial v}\right)\right) \\ &= e^r\{\cos(\theta)(e^r(-\cos(\theta)V_1 + \sin(\theta)V_2)) - \sin(\theta)(e^r(\sin(\theta)V_1 + \cos(\theta)V_2))\} \\ &= -e^{2r}V_1. \end{aligned}$$

Similarly,

$$\begin{aligned} d\nu(V_2) &= d\nu\left(e^r\left(\sin(\theta)\frac{\partial p}{\partial u} + \cos(\theta)\frac{\partial p}{\partial v}\right)\right) \\ &= e^r\{\sin(\theta)(e^r(-\cos(\theta)V_1 + \sin(\theta)V_2)) + \cos(\theta)(e^r(\sin(\theta)V_1 + \cos(\theta)V_2))\} \\ &= e^{2r}V_2. \end{aligned}$$

The previous two equalities show that the vectors V_1 and V_2 define principal directions and that the principal curvatures of the immersion at the point $\rho(u, v)$ are $\pm e^{2r(u, v)}$. This completes the proof of the Theorem. \square

We also have that certain minimal immersions of tori induce solutions of the system (1). The following theorem shows exactly which minimal tori in S^3 are characterized by the integrable system (1).

Theorem 3.12. *Let $\tilde{\rho} : M \rightarrow S^3$ be a minimal immersed torus in S^3 . Using the notation given in the introduction, if for some angle θ and some matrix $B \in so(4)$, $h_\theta = 2f_B$, then, it is possible to choose a covering map $\tau : \mathbb{R}^2 \rightarrow M$, maps $\rho : \mathbb{R}^2 \rightarrow S^3$, $\nu : \mathbb{R}^2 \rightarrow S^3$, $V_1, V_2 : \mathbb{R}^2 \rightarrow S^3$, and a function $r : \mathbb{R}^2 \rightarrow \mathbb{R}$ using Theorem (2.4) and its corollaries, such that*

$$\phi(u, v) = (\rho(u, v), V_1(u, v), V_2(u, v), \nu(u, v), r(u, v), \frac{\partial r}{\partial v}(u, v))$$

is a solution of the system (1) with matrix B and angle θ .

Proof. Using Remark (2.6), we can rotate coordinates so that the maps ρ , ν , V_1 , and V_2 in Theorem (2.4) and Corollaries (2.5) and (2.7) satisfy

$$V_1(u, v) = W_1(\tau(u, v)), \quad V_2(u, v) = W_2(\tau(u, v)), \quad \nu(u, v) = N(\tau(u, v)) \quad \text{and} \quad \alpha = \theta,$$

with $a(\tau(u, v)) = e^{2r}$. Since $\alpha = \theta$,

$$V_1 = e^r(\cos(\theta)\frac{\partial \rho}{\partial u} - \sin(\theta)\frac{\partial \rho}{\partial v}) \quad \text{and} \quad V_2 = e^r(\sin(\theta)\frac{\partial \rho}{\partial u} + \cos(\theta)\frac{\partial \rho}{\partial v}),$$

if $2f_B = h_\theta$, then

$$\begin{aligned} 2\langle B\rho, \nu \rangle &= \cos(\theta)e^{-3r}(e^r(\cos(\theta)\frac{\partial \rho}{\partial u} - \sin(\theta)\frac{\partial \rho}{\partial v}))(e^{2r}) \\ &\quad + \sin(\theta)e^{-3r}(e^r(\sin(\theta)\frac{\partial \rho}{\partial u} + \cos(\theta)\frac{\partial \rho}{\partial v}))(e^{2r}) \\ &= 2\frac{\partial r}{\partial u} \end{aligned}$$

so that

$$(4) \quad 2\langle B\rho, \nu \rangle = 2\frac{\partial r}{\partial u} = h_\theta$$

and, similarly,

$$(5) \quad 2\frac{\partial r}{\partial v} = 2s = h_{\theta+\frac{\pi}{2}}.$$

From the formulas for V_1 and V_2 in Corollary (2.5), we have that

$$\frac{\partial \rho}{\partial u} = e^{-r}(V_1 \cos(\theta) + \sin(\theta)V_2) \quad \text{and} \quad \frac{\partial \rho}{\partial v} = e^{-r}(-V_1 \sin(\theta) + \sin(\theta)V_2),$$

which verify the equations in the integrable system (1). Also, using the equation above and the formula for $\frac{\partial \nu}{\partial u}$ and $\frac{\partial \nu}{\partial v}$ in Theorem (2.4), we get that

$$\frac{\partial \nu}{\partial u} = e^r(-\cos(\theta)V_1 + \sin(\theta)V_2) \quad \text{and} \quad \frac{\partial \nu}{\partial v} = e^r(\sin(\theta)V_1 + \cos(\theta)V_2)$$

Which verify the equations in the integrable system (1). In the same way a direct computation shows that derivatives of $\frac{\partial V_i}{\partial u}$ satisfy the equations of the the system (1). In order to complete the proof of this lemma, let us check the equation for $\frac{\partial s}{\partial v}$. We have that

$$\begin{aligned} \frac{\partial s}{\partial v} &= \frac{\partial^2 r}{\partial v^2} = -2 \sinh(2r) - \frac{\partial^2 r}{\partial u^2} \\ &= -2 \sinh(2r) - \frac{\partial}{\partial u} \langle B\rho, \nu \rangle \\ &= -2 \sinh(2r) - \langle B \frac{\partial \rho}{\partial u}, \nu \rangle - \langle B\rho, \frac{\partial \nu}{\partial u} \rangle \\ &= -\sin(\theta) \langle BV_2, e^{-r}\nu - e^r p \rangle - \cos(\theta) \langle BV_1, e^{-r}\nu + e^r p \rangle, \end{aligned}$$

which verifies the equation in the integrable system (1). The equation for $\frac{\partial s}{\partial u}$ is similar. \square

Remark 3.13. Arguing in the same way we did in the proof of the previous theorem we have that if

$$\phi(u, v) = (\rho(u, v), V_1(u, v), V_2(u, v), \nu(u, v), r(u, v), s(u, v))$$

is a doubly-periodic solution of the integral system (1) and M is the torus $\frac{\mathbb{R}^2}{\sim}$, then,

$$h_\theta([(u, v)]) = 2 \frac{\partial r}{\partial u}(u, v) \quad \text{and} \quad h_{\theta + \frac{\pi}{2}}([(u, v)]) = 2 \frac{\partial r}{\partial v}(u, v) = 2s.$$

Moreover, for any 4×4 skew-symmetric matrix \tilde{B} , $f_{\tilde{B}}([(u, v)]) = \langle \tilde{B}\rho(u, v), \nu(u, v) \rangle$. Also, since ϕ satisfies the integrable system (1), then $h_\theta = 2f_B$.

4. THE LAWSON-HSIANG EXAMPLES

The Lawson-Hsiang tori examples are characterized as those immersed minimal tori in S^3 that are preserved by a 1-parameter group of ambient isometries [3]. This section will show that these examples can be seen, first, as those immersed minimal tori for which there exists a nonzero matrix $B \in so(4)$ such that the function $f_B : M \rightarrow \mathbb{R}$ is identically zero. Then we show that all these examples are included in our new construction. We show that these examples define solutions of the integrable system (1) with data matrix $B \in so(4)$ identically zero. Then, we will prove that if a solution of the integrable system (1) with $B = \mathbf{0}$ defines a minimal torus, then this torus must be one of the examples of Lawson and Hsiang.

Proposition 4.1. *If $\tilde{\rho} : M \rightarrow S^3$ is an immersed closed minimal surface, such that $f_B : M \rightarrow \mathbb{R}$ vanishes for some $B \neq \mathbf{0}$, then $\tilde{\rho}(M)$ is invariant under the group $\{e^{tB} : t \in \mathbb{R}\}$, so that M is one of the examples of Hsiang-Lawson.*

Proof. Let $X : S^3 \rightarrow \mathbb{R}^4$ be the tangent vector field on S^3 given by $X(p) = Bp$. Since $0 = f_B(m) = \langle B\tilde{\rho}(m), N(m) \rangle$, then X induces a unit tangent vector field on M . Therefore the integrals curves of the vector field X that start in $\tilde{\rho}(M)$ remains in $\tilde{\rho}(M)$, i.e. if $\tilde{\rho}(m) \in \tilde{\rho}(M)$ then $e^{tB}\tilde{\rho}(m) \in \tilde{\rho}(M)$. \square

Proposition 4.2. *Let $\tilde{\rho} : M \rightarrow S^3$ be minimal immersion of a torus. If $f_B : M \rightarrow \mathbb{R}$ vanishes, then, for some angle θ , $h_\theta : M \rightarrow \mathbb{R}$ vanishes, and $\tilde{\rho}$ corresponds to a solution of system (1) with $nnt(M) \leq 6$.*

Proof. As in the previous proposition, the vector field $X(m) = B\tilde{\rho}(m)$ defines a tangent vector field on M . Since the function a is invariant under isometries and X is a Killing vector field, then the function $X(a)$ is identically zero. We will prove the proposition by showing that for some fixed angle θ and some fixed real number λ , $X = \lambda a^{-\frac{1}{2}}(\cos(\theta)W_1 + \sin(\theta)W_2)$. Choose maps $\rho, \nu, V_1, V_2 : \mathbb{R}^2 \rightarrow S^3$, a covering $\tau : \mathbb{R}^2 \rightarrow M$ and a function $r : \mathbb{R}^2 \rightarrow \mathbb{R}$ using Theorem (2.4), and its corollaries, such that

$$W_1(\tau(u, v)) = V_1(u, v), \quad W_2(\tau(u, v)) = V_2(u, v) \quad \text{and} \quad N(\tau(u, v)) = \nu(u, v).$$

With this special parameterization of this torus and having in mind that $a(\tau(u, v)) = e^{2r(u, v)}$, we have that $\alpha = 0$ and

$$V_1 = e^r \frac{\partial \rho}{\partial u}, \quad V_2 = e^r \frac{\partial \rho}{\partial v}, \quad W_1(a)(\tau(u, v)) = 2e^{3r(u, v)} \frac{\partial r}{\partial u}(u, v) \quad \text{and} \quad W_2(a)(\tau(u, v)) = 2e^{3r(u, v)} \frac{\partial r}{\partial v}(u, v).$$

Since X is a tangent vector field, $X(\tau(u, v)) = f(u, v)V_1(u, v) + g(u, v)V_2(u, v)$ for two doubly periodic smooth functions $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$. Since, moreover, X is a Killing vector field,

$$\langle \nabla_{W_1} X, W_1 \rangle(\tau(u, v)) = V_1(f)(u, v) - \frac{W_2(a)}{2a}(\tau(u, v))g(u, v) = e^r \left(\frac{\partial f}{\partial u} - g \frac{\partial r}{\partial v} \right) = 0,$$

$$\langle \nabla_{W_2} X, W_2 \rangle(\tau(u, v)) = V_2(g)(u, v) - \frac{W_1(a)}{2a}(\tau(u, v))f(u, v) = e^r \left(\frac{\partial g}{\partial v} - f \frac{\partial r}{\partial u} \right) = 0, \quad \text{and}$$

$$\begin{aligned} (\langle \nabla_{W_1} X, W_2 \rangle + \langle \nabla_{W_2} X, W_1 \rangle)(\tau(u, v)) &= V_1(g)(u, v) + \frac{W_2(a)}{2a}(\tau(u, v))f(u, v) + \\ &\quad V_2(f)(u, v) + \frac{W_1(a)}{2a}(\tau(u, v))g(u, v) \\ &= e^r \left(\frac{\partial g}{\partial u} + f \frac{\partial r}{\partial v} + \frac{\partial f}{\partial v} + g \frac{\partial r}{\partial u} \right) \\ &= 0. \end{aligned}$$

A direct verification gives that the three equations above imply that the function $h(u + iv) = (e^r f)(u, v) + i(e^r g)(u, v)$ is an analytic function. Since h is doubly periodic in \mathbb{R}^2 , and in particular it is bounded, then we get that the function h is constant. We can write this constant as $\lambda \cos(\theta) + i\lambda \sin(\theta)$ with $\lambda \neq 0$.

The rest of the proposition follows from Theorem (3.12), since in this case $nnt(M) \leq 6$. \square

The previous proposition shows that all the examples discussed in [3] are included in the family given by the system (1). More precisely, each one of them is included in one system with $B = \mathbf{0}$. The following proposition shows that if $B = \mathbf{0}$, then every torus in the system (1) is one of the examples in [3]. Recall that our examples characterize those minimal immersions such that $h_\theta = f_B$, therefore, the condition $B = \mathbf{0}$ implies that h_θ vanishes for some angle θ .

Proposition 4.3. *Let $\tilde{\rho} : M \rightarrow S^3$ be a minimal immersion of a torus. If for some θ , $h_\theta : M \rightarrow \mathbb{R}$ vanishes, then f_B vanishes for some nonzero skew-symmetric matrix B .*

Proof. Define the vector field X by $X = a^{-\frac{1}{2}} \cos(\theta)W_1 + a^{-\frac{1}{2}} \sin(\theta)W_2$. The following identities show that X is a Killing vector field on M .

$$\begin{aligned} \langle \nabla_{W_1} X, W_1 \rangle &= -\frac{1}{2}a^{-\frac{3}{2}}W_1(a) \cos(\theta) - a^{-\frac{1}{2}}\frac{1}{2a}W_2(a) \sin(\theta) = -\frac{1}{2a}h_\theta = 0 \\ \langle \nabla_{W_2} X, W_2 \rangle &= -\frac{1}{2}a^{-\frac{3}{2}}W_2(a) \sin(\theta) - a^{-\frac{1}{2}}\frac{1}{2a}W_1(a) \cos(\theta) = -\frac{1}{2a}h_\theta = 0 \\ \langle \nabla_{W_1} X, W_2 \rangle &= -\frac{1}{2}a^{-\frac{3}{2}}W_1(a) \sin(\theta) + a^{-\frac{1}{2}}\frac{1}{2a}W_2(a) \cos(\theta) \\ \langle \nabla_{W_2} X, W_1 \rangle &= -\frac{1}{2}a^{-\frac{3}{2}}W_2(a) \cos(\theta) + a^{-\frac{1}{2}}\frac{1}{2a}W_1(a) \sin(\theta) = -\langle \nabla_{W_1} X, W_2 \rangle. \end{aligned}$$

Therefore the map $\Theta_X(t, \cdot) : M \rightarrow M$ defines a 1-parameter group of isometries in M . By Theorem (2.3), M is invariant under a 1-parameter group of isometries of S^3 , and therefore f_B vanishes for some nonzero $B \in so(4)$. \square

5. MINIMAL SURFACES WITH NATURAL NULLITY LESS THAN 8

The examples of minimal tori found in [3] that are not Clifford tori can be divided into three categories F_1, F_2 and F_3 . The first one, F_1 , consists of the immersions given by

$$\tilde{\rho}(u, v) = (\cos(mx) \cos(y), \sin(mx) \cos(y), \cos(kx) \sin(y), \sin(kx) \sin(y))$$

Where m and k are two relatively-prime positive integers. These examples can be characterized by the property that the principal curvature function $a : M \rightarrow \mathbb{R}$ is constant along a direction that makes a constant angle of $\frac{\pi}{4}$ with respect to one of the principal directions. The second category, F_2 , are the examples found initially by Otsuki [4], and are characterized by the property that the function a is constant along one of the principal directions. The third category, F_3 , are the new examples found in the paper [3] that complete the classification of minimal immersions of tori that are invariant under a group of isometries of S^3 .

Since there is an explicit parameterization $\tilde{\rho}$, these examples explicitly give solutions for the system (1).

Proposition 5.1. *Assume that the variables x and y are related to the variables u and v by the following equations:*

$$u = \int_0^y \sqrt{\frac{mk}{m^2 \cos^2(t) + k^2 \sin^2(t)}} dt \quad \text{and} \quad v = \sqrt{mk}x.$$

For any pair of positive real numbers m and k , the map

$$\phi(u, v) = (\rho(u, v), V_1(u, v), V_2(u, v), \nu(u, v), r(u, v), s(u, v))$$

given by

$$\begin{aligned}
\rho(u, v) &= (\cos(mx) \cos(y), \sin(mx) \cos(y), \cos(kx) \sin(y), \sin(kx) \sin(y)) \\
V_1(u, v) &= \frac{1}{\sqrt{2}}(-\cos(mx) \sin(y), -\sin(mx) \sin(y), \cos(kx) \cos(y), \sin(kx) \cos(y)) \\
&\quad + \frac{1}{\sqrt{2(m^2 \cos^2(y) + k^2 \sin^2(y))}}(-m \sin(mx) \cos(y), m \cos(mx) \cos(y), \\
&\quad \quad \quad -k \sin(kx) \sin(y), k \cos(kx) \sin(y)) \\
V_2(u, v) &= \frac{1}{\sqrt{2}}(\cos(mx) \sin(y), \sin(mx) \sin(y), -\cos(kx) \cos(y), -\sin(kx) \cos(y)) \\
&\quad + \frac{1}{\sqrt{2(m^2 \cos^2(y) + k^2 \sin^2(y))}}(-m \sin(mx) \cos(y), m \cos(mx) \cos(y), \\
&\quad \quad \quad -k \sin(kx) \sin(y), k \cos(kx) \sin(y)) \\
\nu(u, v) &= \sqrt{\frac{1}{m^2 \cos^2(y) + k^2 \sin^2(y)}}(k \sin(mx) \sin(y), -k \cos(mx) \sin(y), \\
&\quad \quad \quad -m \sin(kx) \cos(y), m \cos(kx) \cos(y))
\end{aligned}$$

and

$$r(u, v) = \frac{1}{2} \ln \left(\frac{mk}{m^2 \cos^2(y) + k^2 \sin^2(y)} \right), \quad s(u, v) = 0$$

is a solution of the integrable system (1) with $\theta = -\frac{\pi}{4}$ and

$$B = \begin{pmatrix} 0 & \frac{m^2 - k^2}{k\sqrt{mk}} & 0 & 0 \\ -\frac{m^2 - k^2}{k\sqrt{mk}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly,

Proposition 5.2. *Assume that the variables x and y are related to the variables u and v by the following equations:*

$$v = \int_0^y \sqrt{\frac{mk}{m^2 \cos^2(t) + k^2 \sin^2(t)}} dt \quad \text{and} \quad u = \sqrt{mk}x$$

For any pair of positive real numbers m and k , the map

$$\phi(u, v) = (\rho(u, v), V_1(u, v), V_2(u, v), \nu(u, v), r(u, v), s(u, v))$$

given by

$$\begin{aligned}
\rho(u, v) &= (\cos(mx) \cos(y), \sin(mx) \cos(y), \cos(kx) \sin(y), \sin(kx) \sin(y)) \\
V_1(u, v) &= \frac{1}{\sqrt{2}}(-\cos(mx) \sin(y), -\sin(mx) \sin(y), \cos(kx) \cos(y), \sin(kx) \cos(y)) \\
&\quad + \frac{1}{\sqrt{2(m^2 \cos^2(y) + k^2 \sin^2(y))}}(-m \sin(mx) \cos(y), m \cos(mx) \cos(y), \\
&\quad \quad \quad -k \sin(kx) \sin(y), k \cos(kx) \sin(y)) \\
V_2(u, v) &= \frac{1}{\sqrt{2}}(\cos(mx) \sin(y), \sin(mx) \sin(y), -\cos(kx) \cos(y), -\sin(kx) \cos(y)) \\
&\quad + \frac{1}{\sqrt{2(m^2 \cos^2(y) + k^2 \sin^2(y))}}(-m \sin(mx) \cos(y), m \cos(mx) \cos(y), \\
&\quad \quad \quad -k \sin(kx) \sin(y), k \cos(kx) \sin(y)) \\
\nu(u, v) &= \sqrt{\frac{1}{m^2 \cos^2(y) + k^2 \sin^2(y)}}(k \sin(mx) \sin(y), -k \cos(mx) \sin(y), \\
&\quad \quad \quad -m \sin(kx) \cos(y), m \cos(kx) \cos(y))
\end{aligned}$$

and

$$\begin{aligned}
r(u, v) &= \frac{1}{2} \ln \left(\frac{mk}{m^2 \cos^2(y) + k^2 \sin^2(y)} \right), \\
s(u, v) &= \frac{m^2 - k^2}{\sqrt{mk(m^2 \cos^2(y) + k^2 \sin^2(y))}} \sin(y) \cos(y),
\end{aligned}$$

is a solution of the integrable system (1) with $\theta = -\frac{\pi}{4}$ and matrix $B = \mathbf{0}$.

Remark 5.3. From remark (3.13), for the examples we are studying, those that come from solutions of the integrable system (1), $h_\theta = 2f_B$, therefore $h_\theta \in KS$. Since for any θ , $\text{span}\{h_0, h_{\frac{\pi}{2}}\} = \text{span}\{h_\theta, h_{\theta+\frac{\pi}{2}}\}$, then, if $h_{\theta+\frac{\pi}{2}} = 2f_{\tilde{B}}$ for some 4×4 skew-symmetric matrix \tilde{B} , we have that $\text{span}\{h_0, h_{\frac{\pi}{2}}\} \subset KS$. Notice that, again by remark (3.13), $h_{\theta+\frac{\pi}{2}} = 2\frac{\partial r}{\partial v}(u, v) = s(u, v)$, therefore, in these examples, showing that $\text{span}\{h_0, h_{\frac{\pi}{2}}\} \subset KS$ is equivalent to showing that $s = \frac{\partial r}{\partial v} = f_{\tilde{B}}$ for some 4×4 skew-symmetric matrix \tilde{B} .

From Proposition (5.1) we can deduce that the natural nullity of the family F_1 is 5, since both of the functions h_0 and $h_{\frac{\pi}{2}}$ are contained in the space $\{f_B : B \in so(4)\}$. Notice that by Proposition (4.2), if M is in one of the families F_1 , F_2 , or F_3 , then for some θ the function h_θ vanishes. The question that we will address now is whether the function $h_{(\theta+\frac{\pi}{2})}$, in general, is contained in the set $\{f_B : B \in so(4)\}$, as it is for tori in F_1 . Notice that the question is equivalent to whether the natural nullity of the Lawson-Hsiang examples that are not Clifford tori is equal to 5. The following two theorems resolve the question.

Theorem 5.4. *Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^{18}$ be a solution of integrable system (1), and let $r(u, v) = \phi_{17}(u, v)$ and $s(u, v) = \phi_{18}(u, v)$. Assume that $\phi(0, 0) = x^0 = (e_1, e_2, e_3, e_4, r_0, 0)$ and $\frac{\partial r}{\partial u}(0, 0) = 0$. If*

$$B = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ -b_1 & 0 & b_4 & b_5 \\ -b_2 & -b_4 & 0 & b_6 \\ -b_3 & -b_5 & -b_6 & 0 \end{pmatrix},$$

then, s vanishes if and only if $b_3 = b_4 = 0$ and

- (a) $-e^{r_0} \cos(\theta)b_1 + e^{r_0} \sin(\theta)b_2 + e^{-r_0} \cos(\theta)b_5 + e^{-r_0} \sin(\theta)b_6 = 2 \sinh(2r_0),$
- (b) $-e^{r_0} \sin(\theta)b_1 - e^{r_0} \cos(\theta)b_2 + e^{-r_0} \sin(\theta)b_5 - e^{-r_0} \cos(\theta)b_6 = 0, \text{ and}$
- (c) $-e^{-r_0} \cos(\theta)b_1 - e^{-r_0} \sin(\theta)b_2 + e^{r_0} \cos(\theta)b_5 - e^{r_0} \sin(\theta)b_6 = 0.$

Proof. We will use the integrable system (2) with $\tilde{B} = \mathbf{0}$. Notice that

$$b_3 = -\xi_1(0,0), \quad b_4 = -\xi_4(0,0), \quad b_1 = \xi_2(0,0), \quad b_2 = \xi_3(0,0), \quad b_5 = -\xi_5(0,0), \quad b_6 = -\xi_6(0,0).$$

Assume that $s(u, v) = 0$ for every $(u, v) \in \mathbb{R}^2$. The equation $b_3 = 0$ follows because we are assuming that $\frac{\partial r}{\partial u}(0,0) = \xi_1(0,0) = 0$. Equation (a) in the statement of the theorem follows from the equation $\frac{\partial s}{\partial v}(0,0) = 0$. Equation (b) follows from the equation $\frac{\partial s}{\partial u}(0,0) = 0$. We now prove that $s \equiv 0$ also implies that $b_4 = 0$ and equation (c) in the statement of the theorem.

A direct computation shows the following two equations,

$$\begin{aligned} \frac{\partial^2 s}{\partial v \partial u} &= \xi_1(-2 \cosh(2r) + e^r(\sin(\theta)\xi_3 - \cos(\theta)\xi_2) + e^{-r}(\sin(\theta)\xi_6 + \cos(\theta)\xi_5)) \\ &\quad + s(-e^r(\sin(\theta)\xi_2 + \cos(\theta)\xi_3) + e^{-r}(\sin(\theta)\xi_5 - \cos(\theta)\xi_6)) - 2 \sin(2\theta)\xi_4 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 s}{\partial v^2} &= s(-4 \cosh(2r) + e^r(\sin(\theta)\xi_3 - \cos(\theta)\xi_2) + e^{-r}(\sin(\theta)\xi_6 + \cos(\theta)\xi_5)) \\ &\quad + \xi_1(e^r(\sin(\theta)\xi_2 + \cos(\theta)\xi_3) + e^{-r}(\cos(\theta)\xi_6 - \sin(\theta)\xi_5)) - 2 \cos(2\theta)\xi_4. \end{aligned}$$

From the previous equations we get that $\xi_4(0,0) = -b_4 = 0$ and that $\frac{\partial \xi_4}{\partial v}(0,0) = 0$ because $\xi_1(0,0) = 0$, and

$$\frac{\partial \xi_1}{\partial v}(0,0) = \frac{\partial s}{\partial u}(0,0) = 0.$$

A direct computation shows that the equation (c) in the statement of the theorem is equivalent to the equation $\frac{\partial \xi_4}{\partial v}(0,0) = 0$. So we have shown one implication in the theorem.

We now show the other implication. Assume that we have the equations (a), (b) and (c) on the statement of the theorem, and also $b_4 = b_3 = 0$. These 5 conditions are equivalent to the conditions

$$\xi_1(0,0) = 0, \quad \xi_4(0,0) = 0, \quad \frac{\partial \xi_1}{\partial v}(0,0) = \frac{\partial s}{\partial u}(0,0) = 0, \quad \frac{\partial s}{\partial v}(0,0) = 0, \quad \text{and} \quad \frac{\partial \xi_4}{\partial v}(0,0) = 0.$$

Notice also that by assumption we have that $s(0,0) = 0$. Using the integrable system (2) we can see that the initial conditions above imply that

$$(6) \quad \frac{\partial \xi_i}{\partial u}(0,0) = \frac{\partial \xi_i}{\partial v}(0,0) = 0, \quad \text{for } i = 2, 3, 5, 6,$$

and, also, we can prove by induction that given $n \geq 1$, k and l non-negative integers such that $k + l = n$, there exists a polynomial $P = P(t_1, \dots, t_9)$ such that

$$\frac{\partial^n r}{\partial u^l \partial v^k} = P(e^r, e^{-r}, s, \xi_1, \dots, \xi_6).$$

Along with the equations in (6), these equations imply that

$$\frac{\partial^n s}{\partial u^l \partial v^{k+1}}(0,0) = \frac{\partial(\frac{\partial^n r}{\partial u^l \partial v^k})}{\partial v}(0,0) = \frac{\partial P(e^r, e^{-r}, s, \xi_1, \dots, \xi_6)}{\partial v}(0,0) = 0.$$

In the last equation we have also used the hypothesis that $\frac{\partial \xi_1}{\partial v}(0,0) = \frac{\partial \xi_4}{\partial v}(0,0) = 0$. We should point out that we have used the fact that the function r is real analytic, which follows from the fact that $\Delta r + 2 \sinh(2r) = 0$. \square

Remark 5.5. If $r_0 = 0$, then for any angle θ , the matrices B that satisfy the conditions of the previous theorem form a 2-dimensional subspace of $so(4)$. They have the following form:

$$B_{b_1, b_2} = \begin{pmatrix} 0 & b_1 & b_2 & 0 \\ -b_1 & 0 & 0 & b_1 \\ -b_2 & 0 & 0 & -b_2 \\ 0 & -b_1 & b_2 & 0 \end{pmatrix}.$$

The reason for the existence of this two-dimensional subspace of $so(4)$ is that every Clifford torus M that contains the point $p_0 = (1, 0, 0, 0)$ with tangent space containing the vectors $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$, i.e., with $N(p_0) = (0, 0, 0, \pm 1)$, have the property that $e^{tB_{b_1, b_2}} M = M$. If $r_0 \neq 0$, then, given θ , the matrices B that satisfy the conditions of the previous theorem form a 1-dimensional affine space and are of the form

$$B = B_\theta^1 + \lambda B_\theta^2 = \begin{pmatrix} 0 & -e^{r_0} \cos(\theta) & e^{r_0} \sin(\theta) & 0 \\ e^{r_0} \cos(\theta) & 0 & 0 & -e^{-r_0} \cos(\theta) \\ -e^{r_0} \sin(\theta) & 0 & 0 & -e^{-r_0} \sin(\theta) \\ 0 & e^{-r_0} \cos(\theta) & e^{-r_0} \sin(\theta) & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & e^{-r_0} \sin(\theta) & -e^{-r_0} \cos(\theta) & 0 \\ -e^{-r_0} \sin(\theta) & 0 & 0 & e^{r_0} \sin(\theta) \\ e^{-r_0} \cos(\theta) & 0 & 0 & e^{r_0} \cos(\theta) \\ 0 & -e^{r_0} \sin(\theta) & -e^{r_0} \cos(\theta) & 0 \end{pmatrix}.$$

Theorem 5.6. *If $M \subset S^3$ is an immersed torus invariant under a 1-parametric group of isometries of S^3 , then $nnt(M) = kn(M)$ and therefore the natural nullity $nnt(M) \leq 5$.*

Proof. By Proposition (4.3) we know that for some angle θ , $(\cos(\theta)V_1 + \sin(\theta)V_2)(a) = 0$ where $a : M \rightarrow \mathbf{R}$ is a positive function such that the principal curvatures of M at p are $\pm a(p)$. Without loss of generality, we will assume that

$$e_1 \in M, V_1(e_1) = e_2, V_2(e_1) = e_3, \nu(e_1) = e_2, \ln a(e_1) = 2r_0, \text{ and } \nabla a(e_1) = \mathbf{0}.$$

Therefore, M defines a solution of the system (1) associated with the matrix $B = \mathbf{0}$ and θ . Call this solution $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^{18}$. Without loss of generality we can assume that $\phi(0, 0) = (e_1, e_2, e_3, e_4, r_0, 0)$.

Define $\tilde{\phi}$ to be the solution of the system (1) associated with a matrix $B = \{b_{ij}\}$ that satisfies the conditions in the previous lemma and $\tilde{\theta} = \theta - \frac{\pi}{2}$. Moreover we will take the initial solution that satisfies

$$\tilde{\phi}(0, 0) = (e_1, e_2, e_3, e_4, r_0, 0).$$

Now consider the map $\hat{\phi} : \mathbb{R}^2 \rightarrow \mathbb{R}^{18}$ given by

$$\begin{aligned} \hat{\phi}(u, v) &= (\hat{\rho}(u, v), \hat{V}_1(u, v), \hat{V}_2(u, v), \hat{\nu}(u, v), \hat{r}(u, v), \hat{s}(u, v)) \\ &= (\tilde{\rho}(-v, u), \tilde{V}_1(-v, u), \tilde{V}_2(-v, u), \tilde{\nu}(-v, u), \tilde{r}(-v, u), -\langle B\tilde{\rho}, \tilde{\nu} \rangle), \end{aligned}$$

where

$$\tilde{\phi}(\tilde{u}, \tilde{v}) = (\tilde{\rho}(\tilde{u}, \tilde{v}), \tilde{V}_1(\tilde{u}, \tilde{v}), \tilde{V}_2(\tilde{u}, \tilde{v}), \tilde{\nu}(\tilde{u}, \tilde{v}), \tilde{r}(\tilde{u}, \tilde{v}), \tilde{s}(\tilde{u}, \tilde{v})).$$

It is clear that $\hat{\phi}(0, 0) = (e_1, e_2, e_3, e_4, r_0, 0)$. Notice that, by the way B was chosen, we have that $\tilde{s} = 0$ for every $(\tilde{u}, \tilde{v}) \in \mathbb{R}^2$. Also, a direct computation shows that $\hat{\phi}$ is a solution of the system (1) with $B = \mathbf{0}$ and the angle θ , therefore, $\hat{\phi}(u, v) = \phi(u, v)$, and so

$$\frac{\partial r}{\partial v} = -\frac{\partial \tilde{r}}{\partial \tilde{u}} = -\langle B\rho, \nu \rangle.$$

This equality is equivalent to the fact that $\sin(\theta)u_1 - \cos(\theta)u_2 = f_B$, where the functions $u_1 = h_0$, $u_2 = h_{\pi/2}$, and f_B are defined in the first section. This last equation implies that $h_{\theta+\frac{\pi}{2}} = -f_B$, therefore, h_θ , which is identically zero, and $h_{\theta+\frac{\pi}{2}}$ are functions in $\{f_C : C \in so(4)\}$. Then, both functions u_1 and u_2 are also generated by the functions in the set $\{f_C : C \in so(4)\}$, i.e., the natural nullity is 5. Recall that the space $\{u_C : C \in so(4)\}$ is 5-dimensional for any torus invariant under a 1-parameter group of isometries in S^3 . \square

Corollary 5.7. *Let M be torus with natural nullity 5, and let $\theta \in \mathbb{R}$ and $B \in so(4)$ be such that $h_\theta = 0$ and $h_{(\theta-\frac{\pi}{4})} = f_B$. If B_θ^1 and B_θ^2 are defined as in the Remark (5.5), then*

$$B = B_\theta^1 + \lambda_0 B_\theta^2 \quad \text{for some } \lambda_0 \quad \text{and} \quad e^{\lambda B_\theta^3} M = M \quad \text{for every } \lambda \in \mathbb{R}.$$

Proof. If $nnt(M) = 5$, then certainly there is a nonzero $B \in so(4)$ so that $f_B = 0$. Proposition (4.2) then implies the existence of a θ for which $h_\theta = 0$. $B = B_\theta^1 + \lambda_0 B_\theta^2$ follows from the previous theorem and the Remark (5.5). The second part of the corollary follows from the fact that in the argument used to prove Theorem (5.6), we can choose any B that satisfies the conditions of Theorem (5.4), in particular if we can also choose $B = B_\theta^1 + (\lambda_0 + 1)B_\theta^2$ we will get that

$$\frac{\partial r}{\partial v} = \langle (B_\theta^1 + \lambda_0 B_\theta^2)p, \nu \rangle = \langle (B_\theta^1 + (\lambda_0 + 1)B_\theta^2)p, \nu \rangle.$$

This equation implies that $f_{B_\theta^2} = 0$. The corollary follows by the Proposition (4.1). \square

Corollary 5.8. *If M is a minimal immersed torus in S^3 , then $nnt(M) \leq 5$ if and only if M is one of the examples of Hsiang and Lawson.*

Proof. If M has $nnt(M) \leq 5$, then $kn(M) \leq 5$. Therefore, for some nonzero skew-symmetric matrix B , f_B vanishes. By Proposition(4.1), M will be invariant under a 1-parameter subgroup of the rigid motions of S^3 , which, following [3], implies that M is one of Hsiang and Lawson's examples. On the other hand, since any of the Hsiang-Lawson examples are preserved by a one-parameter subgroup of $SO(4)$, there is a $B \in so(4)$ for which $f_B = 0$. Then Theorem (5.6) implies $nnt(M) \leq 5$. \square

Theorem (5.6) and Corollary (5.7) address the question of the injectivity of the function that sends any pair (θ, B, r_0) to the minimal immersion of the plane with initial conditions $(e_1, e_2, e_3, e_4, r_0, 0)$. The following result is in the same direction.

Proposition 5.9. *If for some $\theta_2 \neq \theta_1 + n\pi$ for any integer n , $h_{\theta_1} = f_{B_1}$ and $h_{\theta_2} = f_{B_2}$, then the natural nullity of M is less than 7.*

Proof. The equations in the Proposition implies that the space $\{\lambda h_\theta : \lambda, \theta \in \mathbb{R}\}$ is a subset of the space $\{f_B : B \in so(4)\}$ which has dimension at most 6. The proposition then follows. \square

Lemma 5.10. *If a solution of (1) satisfies $r(0, 0) = r_0$, $\xi_1(0, 0) = s(0, 0) = \xi_4(0, 0) = 0$, then $r(u, v) = r(-u, -v)$.*

Proof. A direct computation using the system (2) shows that the conditions $\xi_1(0, 0) = s(0, 0) = \xi_4(0, 0) = 0$ give

$$\frac{\partial \xi_i}{\partial u}(0, 0) = \frac{\partial \xi_i}{\partial v}(0, 0) = 0 \quad \text{for } i = 2, 3, 5, 6.$$

Let $C^\omega(\mathbb{R}^2)$ be the set of analytic functions on \mathbb{R}^2 and let P_0 be the ideal of $C^\omega(\mathbb{R}^2)$ generated by the functions $\{e^r, e^{-r}, \xi_2, \xi_3, \xi_5, \xi_6\}$. Given a nonnegative integer k , define P_k as the set of functions in $C^\omega(\mathbb{R}^2)$ that can be written as a homogeneous polynomial of degree k in the variables s, ξ_1 and ξ_4 with coefficients in P_0 . A direct computation using the system (2) give us that if $f \in P_0$, then $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are in P_1 . In the same way, if $f \in P_k$ then $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are in $P_{k+1} + P_{k-1}$. Now with

these observations in mind, we proceed to show that the function r satisfies $r(u, v) = r(-u, -v)$, by showing that all the partial derivatives of odd order of the function r vanish at $(0, 0)$. To achieve this we first notice that the first derivatives of r , the functions ξ_1 and s vanish at $(0, 0)$. Then, notice that the second derivatives of r , i.e. the first derivatives of s and ξ_1 , are functions in P_0 . The last statement implies that the third derivatives of r are in P_1 and therefore they vanish at $(0, 0)$. Once we know that the third derivatives of r are in P_1 we get that the fourth derivatives of r are in $P_0 + P_2$. If we continue with this process we notice that if k is a positive even integer, then the k -th derivatives of r are functions in $P_0 + P_2 + \cdots + P_{k-2}$, and in the case that k is a odd integer greater than 1, then, the k -th derivatives of r are in $P_1 + P_3 + \cdots + P_{k-2}$. Now, since $\xi_1(0, 0) = s(0, 0) = \xi_4(0, 0) = 0$, the odd derivatives of the function r vanish at $(0, 0)$. \square

Theorem 5.11. *Let M be a minimal torus immersed in S^3 . If $nnt(M) \leq 6$, then the group of isometries of M is not trivial.*

Proof. Unless there is some nonzero $B \in so(4)$ for which $f_B = 0$, in which case Proposition (4.1) implies the existence of a one-parameter group of isometries of S^3 which restrict to isometries of M , then $nnt(M) \leq 6$ implies that the span of $\{u_1, u_2\}$, $u_1 := a^{-\frac{3}{2}}W_1(a) = h_0$ and $u_2 := a^{-\frac{3}{2}}W_2(a) = h_{\frac{\pi}{2}}$, will be contained in the span of $\{f_B | B \in so(4)\}$. Since then $u_1 = 2f_B$ for some $B \in so(4)$, then M defines a solution ϕ of the system (1) associated with the matrix B and with $\theta = 0$. The condition $u_2 = 2f_{\tilde{B}}$ implies by Remark(3.13) that $s = \tilde{\xi}_1$, for the system (2) associated with the matrices B, \tilde{B} and $\theta = 0$. As before, we will assume that $\xi_1(0, 0) = s(0, 0) = 0$ and $r(0, 0) = r_0$. Define the function $f = s - \tilde{\xi}_1$. The hypothesis in the theorem is equivalent to the condition that f is identically zero, in particular, $\tilde{\xi}_1(0, 0) = 0$, since $f(0, 0) = 0$. The theorem is a consequence of the previous lemma and will follow by showing that $\xi_4(0, 0) = 0$. A direct computation shows that

$$\frac{\partial f}{\partial u} = e^{-r}\xi_6 - e^r\xi_3 - e^{-r}\tilde{\xi}_5 - e^r\tilde{\xi}_2$$

and

$$\begin{aligned} \frac{\partial^2 f}{\partial u^2} &= \xi_1(-e^{-r}\xi_6 - e^r\xi_3 + e^{-r}\tilde{\xi}_5 - e^r\tilde{\xi}_2) \\ &\quad + e^{-r}(-s\xi_5 + e^r\xi_4) - e^r(-s\xi_2 - e^{-r}\xi_4) \\ &\quad - e^{-r}(-s\xi_5 + e^r\xi_4) - e^r(-s\xi_2 - e^{-r}\xi_4) \\ &\quad - e^r(s\tilde{\xi}_6 - e^{-r}\tilde{\xi}_1) - e^r(s\tilde{\xi}_3 - e^{-r}\tilde{\xi}_1) \\ &= \xi_1(-e^{-r}\xi_6 - e^r\xi_3 - e^{-r}\tilde{\xi}_5 - e^r\tilde{\xi}_2) \\ &\quad + s(-e^{-r}\xi_5 + e^r\xi_2 - e^{-r}\tilde{\xi}_6 - e^r\tilde{\xi}_3) \\ &\quad + 2\xi_4 + 2\cosh(2r)\xi_1. \end{aligned}$$

From the last equation, using the fact that $s(0, 0) = \xi_1(0, 0) = \tilde{\xi}_1(0, 0)$ and $\frac{\partial^2 f}{\partial u^2} = 0$, we conclude that $\xi_4(0, 0) = 0$, which implies, by the previous lemma, that $r(u, v) = r(-u, -v)$. To finish the proof of the theorem, we notice that the function $A(u, v) = -(u, v)$ preserves the lattice in \mathbb{R}^2 given by the double-periodicity of the function ϕ and therefore induces a function in the torus $\tau(\mathbb{R}^2) = M$, since the first fundamental form of M in the coordinates u and v is $ce^{-2r}(du^2 + dv^2)$ where c is a positive constant, then, this function from M to M induced by A is an isometry. \square

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Current address: Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015-3174

Current address: Department of Mathematics, Central Connecticut State University, New Britain, CT 06050,

E-mail address: david.johnson@lehigh.edu, osperdom@univalle.edu.co